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Unitarity Limits on the Mass and Radius of Dark Matter Particles

KIM GRIEST

*Center for Particle Astrophysics, University
of California, Berkeley, CA 94720*

and

MARC KAMIONKOWSKI

*Physics Department, Enrico Fermi Institute,
The University of Chicago, Chicago, IL 60637*

and

*NASA/Fermilab Astrophysics Center, Fermi
National Accelerator Laboratory, Batavia, IL 60510.*

ABSTRACT

Using partial wave unitarity and the observed density of the Universe, we show that any stable elementary particle which was once in thermal equilibrium cannot have a mass greater than 340 TeV. An extended object which was once in thermal equilibrium cannot have a radius less than 7.5×10^{-7} F. A lower limit to the relic abundance of such particles is also found.



The idea that the dark matter known to exist in galactic halos consists of some, as yet undiscovered, stable massive particle has received a great deal of attention in recent years. Dozens of particle candidates have been suggested and new ones are constantly being invented. Most of these dark matter candidates have relic abundances which are calculated in the “Lee-Weinberg” manner and model parameters are typically adjusted to allow their density today to be near critical density, $\Omega_X \approx 1$. In this letter we wish to point out, that for almost any such particle which was once in thermal equilibrium and has an abundance determined in this way, partial wave unitarity of the S matrix bounds the annihilation cross section in the early universe, which in turn bounds the relic abundance and the mass of the particle. In general we find that stable elementary particles with masses greater than around 340 TeV are very likely excluded. Extended objects¹ with radii less than 7.5×10^{-7} F are also very likely excluded.

As an application of these limits, we note that the claim of Enqvist, *et al.*,² that there is no upper limit from cosmology on the mass of a stable Dirac neutrino cannot be true. While the mass upper limit we find is not rigorous and rather high, we still feel it may be of some interest because of its general nature.

The relic abundance of a particle species, X , which was once in thermal and chemical equilibrium, is determined by its total thermally averaged annihilation cross section $\langle \sigma(X\bar{X} \rightarrow \text{all})v_{\text{rel}} \rangle$ at freeze-out. At high temperatures the number density of X ’s is roughly the same as the number density of photons, but as the temperature drops below the mass of the X , their number density drops exponentially. This continues until the total annihilation cross section is no

longer large enough to maintain equilibrium and the X number density then “freezes-out”. The number density today is given roughly by³

$$\Omega_X h^2 = \frac{1.07 \times 10^9 (n+1) x_f^{n+1} \text{GeV}^{-1}}{g_*^{1/2} m_{pl} \langle \sigma v_{\text{rel}} \rangle'_f} \approx \frac{3 \times 10^{-27} \text{cm}^3/\text{sec}}{\langle \sigma v_{\text{rel}} \rangle_f}, \quad (1)$$

where $\Omega_X = \rho_X/\rho_{\text{crit}}$ is the present average density of X ’s divided by the critical density, $\frac{1}{2} \leq h \leq 1$ is the Hubble constant in units of 100 km/sec/Mpc, $x_f = m_X/T_f$, T_f is the freeze-out temperature, $g_* \approx 107$ is the effective number of degrees of freedom at T_f , and $m_{pl} = 1.22 \times 10^{19}$ GeV.

Since the annihilation is non-relativistic (the velocity of the annihilating X particle is $v \approx \frac{1}{4} \ll 1$ at freeze-out) one can expand the cross section in powers of $v^2 \equiv v_{\text{rel}}^2/4$ and keep only the first (or first two) terms. In thermal averaging one replaces $\langle v_{\text{rel}}^2 \rangle = 6/x_f$ and so in eq. (1) the cross section is written $\langle \sigma v_{\text{rel}} \rangle = \langle \sigma v_{\text{rel}} \rangle' x^{-n}$, where n parameterizes the dependence of the cross section on x . The freeze-out temperature is given roughly by³

$$x_f = \ln B - (n + \frac{1}{2}) \ln \ln B, \quad (2)$$

where $B = .038 g m_{pl} m_X \langle \sigma v_{\text{rel}} \rangle' / \sqrt{g_*}$ and g is the number of degrees of freedom of the X particle. Typically $x_f \approx 25$ corresponding to $v_{\text{rel}}^2/4 \approx 1/16$ at freeze-out. Please note that v_{rel} is not really a velocity, but is related to the flux factor. It is defined as $v_{\text{rel}} = 2v$, and so $0 \leq v_{\text{rel}}^2/4 \leq 1$.

From eq. (1) we see that if $\langle \sigma v_{\text{rel}} \rangle_f \ll 3 \times 10^{-27} \text{cm}^3/\text{sec}$, then $\Omega_X h^2 \gg 1$, which would be inconsistent with the “observation”, $\Omega_{\text{tot}} h^2 \leq 1$. Any particle model which predicts an annihilation cross section smaller than this critical value

at $v_{\text{rel}}^2/4 \approx \frac{1}{16}$ is therefore inconsistent with cosmology.⁴ We will now show that partial wave unitarity provides a maximum possible cross section and therefore a minimum possible $\Omega_X h^2$. Extremely massive elementary particles and very small extended objects violate these bounds and therefore are inconsistent with cosmology.

Consider the process $a + b \rightarrow c + d$ and the scattering matrix

$$\langle f|S|i\rangle = \langle f|i\rangle + i(2\pi)^4 \delta^4(P_f - P_i) \langle f|T|i\rangle \quad (3)$$

where $P_i = p_a + p_b$ and $P_f = p_c + p_d$. The T matrix can be expanded in partial waves using the helicity formalism⁵

$$\langle \lambda_c \lambda_d | T(s, \Omega) | \lambda_a \lambda_b \rangle = 8\pi s^{1/2} e^{i\phi(\lambda - \lambda')} \sum_J (2J + 1) d_{\lambda\lambda'}^J(\theta) \langle \lambda_c \lambda_d | T_J(s) | \lambda_a \lambda_b \rangle, \quad (4)$$

where $\lambda_a, \dots, \lambda_d$ are the helicities of particle a, \dots, d . $\lambda = \lambda_a - \lambda_b$, $\lambda' = \lambda_c - \lambda_d$, s is the Mandelstam variable, $\Omega = (\theta, \phi)$ is the center-of-mass scattering angle, and $d_{\lambda\lambda'}^J$ are the Wigner functions.

Using matrix notation⁶ $\langle \lambda_c \lambda_d | T_J(s) | \lambda_a \lambda_b \rangle = (T_J)_{if}$ and $\tilde{p}_k = \text{diag}(p_1, p_2, \dots)$, where p_k is the center of mass three-momentum of particle system i, f , etc. partial wave unitarity of the S matrix can be written⁶

$$T_J - T_J^\dagger = 2iT_J \tilde{p} T_J^\dagger. \quad (5)$$

Defining $S_J = 1 + 2i\tilde{p}^{1/2} T_J \tilde{p}^{1/2}$, we see that partial wave unitarity can also be

written $S_J S_J^\dagger = 1$ or

$$|S_{el,J}|^2 + \sum_f |S_{i \neq f,J}|^2 = 1, \quad (6)$$

where $S_{el,J}$ stands for the elastic channel, $i = f$. The next step is to define $S_{el,J} = \eta_J e^{2i\delta_J}$, where δ_J is a real phase shift and η_J is an inelasticity factor, $0 \leq \eta_J \leq 1$. Then $|S_{el,J}|^2 = \eta_J^2$, and $\sum_f |S_{i \neq f,J}|^2 = 1 - \eta_J^2$. Finally, using $T_{el,J} = (S_{el,J} - 1)/(2ip)$ and $T_{f \neq i,J} = S_{f \neq i,J}/(2i\sqrt{p_i p_f})$, and the standard formula for the unpolarized cross section in terms of partial waves $\sigma = \sum \sigma_J$, where

$$\sigma_J = \frac{4\pi(2J+1)}{(2s_a+1)(2s_b+1)} \sum_\lambda \sum_f \frac{p_f}{p_i} |T_{if,J}|^2, \quad (7)$$

we find the result of Pilkuhn⁶

$$\sigma_{r,J} = 4\pi \frac{(2J+1)}{(2s_a+1)(2s_b+1)} \sum_\lambda \sum_{f \neq i} \frac{p_f}{p_i} |T_{if,J}|^2 = \frac{\pi(2J+1)(1-\eta_J^2)}{p_i^2}. \quad (8)$$

Here $\sigma_{r,J}$ is the “reaction” cross section, that is, the total cross section minus the elastic piece. It has a maximum when $\eta_J = 0$, so we conclude that

$$\sigma_J(a+b \rightarrow c+d) \leq \frac{\pi(2J+1)}{p_i^2}. \quad (9)$$

In the early Universe,

$$p_i^2 = E^2 - m_X^2 = \frac{m_X^2 v_{\text{rel}}^2}{4(1 - v_{\text{rel}}^2/4)} \approx m_X^2 v_{\text{rel}}^2/4,$$

so $\sigma_J v_{\text{rel}} \leq (\sigma_J)_{\text{max}} v_{\text{rel}}$, where

$$(\sigma_J)_{\text{max}} v_{\text{rel}} \approx \frac{4\pi(2J+1)}{m_X^2 v_{\text{rel}}} \approx \frac{3 \times 10^{-22} (2J+1) \text{cm}^3/\text{sec}}{(m_X/\text{TeV})^2}. \quad (10)$$

In order to apply the limits of eq. (10) to the annihilation in the early Universe we need to decide which partial waves contribute. After summing over helicities,

the angular dependence, $\cos \theta$, which indicates the partial wave, enters the cross section only through the Mandelstam variable

$$\begin{aligned} t &= m_a^2 + m_c^2 - 2E_a E_c + 2p_c p_a \cos \theta \\ &= m_a^2 + m_c^2 - 2E_a E_c + 2p_c \cos \theta \, m_X v_{\text{rel}}/2 + O(v_{\text{rel}}^2/4). \end{aligned} \tag{11}$$

So there is a factor of v_{rel} appearing with every factor of $\cos \theta$. In the expansion of the annihilation cross section in powers of $v_{\text{rel}}^2/4 \approx 1/16$, the lowest order term $O((v_{\text{rel}}^2/4)^0)$, therefore has no angular dependence and must be a $J = 0$ partial wave. The $J = 1$ partial wave is smaller by a factor $v_{\text{rel}}^2/4$, and the higher partial waves are further suppressed. In fact, since partial wave unitarity must hold for any value of $v_{\text{rel}}^2/4$, and when $v_{\text{rel}}^2/4$ increases, the maximum cross section, eq. (10), decreases, the $J = 0$ bound, taken when $v_{\text{rel}}^2/4 \approx 1/16$, is not as stringent as possible. The $J = 1$ maximum cross section also decreases for larger $v_{\text{rel}}^2/4$, and more importantly, the term in the actual cross section of order $v_{\text{rel}}^2/4$, increases. If the $J = 1$ bound is satisfied for a larger value of $v_{\text{rel}}^2/4$, for instance $v_{\text{rel}}^2/4 \approx 1/2$, then the $J = 1$ partial wave is below the bound by a factor of $8^{-3/2} \approx .04$ by freeze-out. We conclude that it is more than adequate to use only the $J = 0$ partial wave in finding a bound.

Now we use eqs. (1), (2), and (10) to bound $\Omega_X h^2$ and m_X . Including only the $n = 0$ part of the cross section and replacing $v_{\text{rel}} = \sqrt{6/x_f}$, we find that

$$\Omega_X h^2 \geq 1.7 \times 10^{-6} \sqrt{x_f} (m_X/\text{TeV})^2 \tag{12}$$

for a Majorana fermion with $g = 2$. For a Dirac fermion, $\Omega_X h^2$ is a factor of two

larger. Now using $\Omega_X h^2 \geq 1$, we find the mass limit

$$m_X \leq 340 \text{ TeV}, \quad (13)$$

and $x_f \approx 28$. Eq. (13) was found for a Majorana fermion. The limit for a scalar particle is similar, while for a Dirac fermion is about a factor of $\sqrt{2}$ smaller, that is, $m_X \leq 240 \text{ TeV}$. This is the main result of this letter.

Another, more conservative, way of finding the mass bound is to assume that the cross section, eq. (10), holds throughout the period of annihilation and freeze-out. In this case, the v_{rel}^{-1} factor affects the thermal averaging and the integration from freeze-out to today. The thermally averaged maximum cross section becomes

$$\langle (\sigma_J)_{\text{max}} v_{\text{rel}} \rangle \approx \frac{4\pi(2J+1)}{m_X^2} \left(\frac{x^{1/2}}{\sqrt{\pi}} \right), \quad (14)$$

and the relic abundance is given by eq. (1), with $n = -1/2$,

$$\Omega_X h^2 \geq \frac{6.0 \times 10^{-7}}{(2J+1)} x_f^{1/2} (m_X/\text{TeV})^2. \quad (15)$$

The freeze-out temperature is the same as before with $\langle \sigma v_{\text{rel}} \rangle'$ multiplied by a factor of $(6/\pi)^{1/2}$. (We set $n = -1/2$ in eq. (2), both now and before, since the $x_f^{1/2}$ here is just an algebraic factor.) Using these formulas, the mass limit becomes $m_X < 550 \text{ TeV}$. This is probably an overly conservative bound since one does not expect $\sigma v_{\text{rel}} \propto v_{\text{rel}}^{-1}$ for annihilation channels in a nonrelativistic expansion.

However, we do not claim that the derivation leading to eq. (13) is rigorous, or that exceptions cannot occur. For example, elastic scattering via t -channel

exchange of a massless particle gives rise to a term in the matrix element proportional to $t^{-1} \propto v_{\text{rel}}^{-2}(1 - \cos \theta)^{-1}$. Naively expanding this would suggest that all partial waves contribute to the term of lowest order in $v_{\text{rel}}^2/4$. The problem, in this case, is that we are outside the Lehmann ellipse of convergence, and the partial wave expansion not valid. Fortunately, in annihilation, the mass of the annihilation product must be less than m_X , and the partial wave expansion converges, giving nicely the results we claim above. Another possible exception, which we do not consider very likely, is that the coefficients of the partial wave expansion contain factors of $(s - 4m_X^2)^{-1} \propto v_{\text{rel}}^{-2}$, in just such a way as to cancel the v_{rel}^2 factors associated with the $\cos^2 \theta$ factors. For elastic scattering, it can be proved that this cannot occur (Ref. 6, page 291), but we have been unable to complete the proof for the inelastic case. This may be related to the possibility of s -channel poles, which can cause another possible exception to our limit. A factor of $(s - m_i^2)^{-1}$, with $m_i = 2m_X$ will give an additional factor of v_{rel}^{-2} , in which case partial waves up to $J = 2$ need to be included in our maximum cross section, and the mass limit weakens. However, we feel that such a pole is unlikely. It requires not only an exchange particle of precisely twice the mass of the X , but also that the exchange particle be nearly stable. The width of the exchange particle will dominate the pole unless it is very small, and since the exchanged particle is more massive than the X , and has decay channels into lighter particles, we consider this possibility remote.

We note that the mass limit, eq. (13), involves a mass somewhat higher than typically considered in particle dark matter model building. But since the bound is so general we feel it may be of some use. As an example, we can immediately

apply it to candidates which appear in the literature, such as the Dirac neutrino.

The Dirac neutrino was the first dark matter particle considered and very early Zeldovich⁷ claimed a range of neutrino masses, $3 \text{ GeV} < m_\nu < 3 \text{ TeV}$, as being cosmologically acceptable. His upper bound was based on neutrino annihilation into fermions through Z boson exchange. This cross section is proportional to m_ν^{-2} in the high mass limit. However, Enqvist, Kainulainen and Maalampi² noted that the W^+W^- channel, among others, open up for very massive neutrinos, and that these new channels dominate the cross section in the high mass limit. In fact, they claimed that because the cross section keeps growing as m_ν increases, there is *no* upper limit from cosmology on Dirac neutrino masses. This claim is clearly contradicted by the bound for Dirac fermions given just after eq. (13). Yet we do not believe that that the cross section of Enqvist, *et al.* is in error. We believe that the solution⁸ to this puzzle is that in the Standard Model, where neutrinos get their mass by the Higgs mechanism, as $m_\nu \rightarrow \infty$, the neutrino Yukawa coupling becomes large and perturbation theory breaks down. Another way of saying this is that the higher loop corrections become important in this limit and the tree level calculation of Enqvist, *et al.* is not applicable. In fact, by using unitarity to bound the largest eigenvalue of the scattering matrix, Chanowitz, Furman and Hinchliffe,⁹ showed that the breakdown of perturbation theory occurs at around $m_\nu \approx 1 \text{ TeV}$, far below the limit we set. The breakdown of perturbation theory suggests that the neutrino becomes “strongly interacting” and could not exist as a free, stable state. In this case, the annihilation cross section would be governed by different physics, if the theory made sense at all. If, on the other hand, the neutrino for some reason stays “elementary”, we argue that

our limit applies, giving an upper limit on the neutrino mass from cosmology, just as Zeldovich originally suggested (though at a different value).

Finally, we should comment on the applicability of these bounds to extended objects. For these objects, higher partial waves will generally contribute to the nonrelativistic cross section, and the cosmological mass bound, eq. (13) does not apply; however, partial wave unitarity may still be used to limit the total annihilation cross section, and cosmology provides a constraint on the size of such objects. Consider an extended object with spin 0 and radius R_X . The highest partial wave that can contribute to the particle-antiparticle collision is roughly $J_{max} = 2m_X v_{rel} R_X$, resulting in a maximum total cross section,

$$(\sigma v_{rel})_{max} \approx \frac{4\pi}{m_X^2 v_{rel}} \sum_{J=0}^{J_{max}} (2J+1) \approx 16\pi R_X^2 v_{rel}, \quad (16)$$

four times the geometric cross section. Using eqs. (1), (2), and (16), we can now bound $\Omega_X h^2$ and R_X . We find that

$$\Omega_X h^2 \geq \frac{4 \times 10^{-15} x_f^{\frac{3}{2}}}{(R_X/F)^2} \quad (17)$$

which leads to the bound

$$R_X \geq 7.5 \times 10^{-7} \text{ F}. \quad (18)$$

Here we used $x_f = 27$ which was obtained from eq. (2) using $m_X = 1000 \text{ TeV}$; the radius limit, eq. (18), varies only logarithmically with m_X . The limit for spin $\frac{1}{2}$ particles is more stringent by a factor of $\sqrt{2}$.

We point out that eq. (16) is valid only if $J_{max} \gg 1$. On the other hand, if $J_{max} \ll 1$, the cross section is bound by eq. (9) with $J=0$. Since freeze-out occurs when $v_{rel} \approx \frac{1}{2}$, eq. (18) is reliable only when $R_X \gg 1/m_X$, while an object with $R_X \ll 1/m_X$ must be considered point-like and its mass limited by eq. (13). Furthermore, we note that there is no major discontinuity in the overlap region, $R_X \sim 1/m_X$, since the mass limit for point-like particles, eq. (13), is very nearly that which we would have obtained from the radius limit, eq. (18), had we used the Compton wavelength of the particle for R_X .

In conclusion, although our derivation is not rigorous, and exceptions may exist, we believe that the limit on mass, eq. (13), radius, eq. (18), and relic abundance, eq. (12) should apply to almost any dark matter particle which was once in thermal and chemical equilibrium.

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8. We would like to acknowledge David Seckel for pointing this puzzle out to us, and Scott Willenbrock for indicating its solution. Actually, Dolgov and Zeldovich⁷ give a somewhat similar caveat to ours, and Enqvist, *et al.*² also state that their cross section is suspect as m_ν increases for the same reason.
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